

NOTES ON TRANSFORMATIONS IN INTEGRABLE GEOMETRY

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PROSPECTUS

Roughly speaking, a differential-geometric system, be it smooth, discrete or semi-discrete, is integrable if it has some or all of the following properties:

1. an infinite-dimensional symmetry group.
2. explicit solutions.
3. algebro-geometric solutions via spectral curves and/or theta functions.

In these talks, I shall focus on a manifestation of the first item: transformations whereby new solutions are constructed from old. The theory applies in many situations including:

- surfaces in \mathbb{R}^3 with constant mean or Gauss curvature [1, 4, 24] or, more generally, linear Weingarten surfaces in 3-dimensional spaces forms.
- (constrained) Willmore surfaces in S^n [15].
- projective minimal and Lie minimal surfaces in \mathbb{P}^3 and S^3 respectively [12, 18].
- affine spheres [7, 23].
- harmonic maps of a surface into a pseudo-Riemannian symmetric space [28, 29]: this includes many of the preceding examples via some form of Gauss map construction.
- isothermic surfaces in S^n [2, 3, 9, 16, 21, 26] or, more generally, isothermic submanifolds in symmetric R -spaces [14].
- Möbius flat submanifolds of S^n [10, 11]: these include Guichard surfaces and conformally flat submanifolds with flat normal bundles, in particular, conformally flat hypersurfaces.
- omega surfaces in Lie sphere geometry [25].
- curved flats in pseudo-Riemannian symmetric spaces: these are related to the last four items.
- self-dual Yang–Mills fields [19]: many of our low-dimensional examples are dimensional reductions of these [30].

In these lectures, I shall discuss this theory via two examples: *K-surfaces* (surfaces in \mathbb{R}^3 of constant Gauss curvature) and *isothermic surfaces*. In both cases, I will emphasise:

- the geometry of transformations
- a gauge-theoretic approach of wide applicability.

1. *K*-SURFACES

1.1. **Classical surface geometry.** Let $f : \Sigma^2 \rightarrow \mathbb{R}^3$ be an immersion with Gauss map $N : \Sigma \rightarrow S^2$. Thus:

$$N \cdot df = 0.$$

These yield three invariant quadratic forms:

$$\text{I} := df \cdot df$$

$$\text{II} := -df \cdot dN$$

$$\text{III} := dN \cdot dN$$

and the famous theorem of Bonnet says that the first two determine f up to a rigid motion.

Lowering an index on II gives the *shape operator* $S := -(df)^{-1} \circ dN$, a symmetric (with respect to I) endomorphism on $T\Sigma$. The shape operator has eigenvalues κ_1, κ_2 , the *principal curvatures* from which

the mean curvature H and the Gauss curvature K are given by

$$\begin{aligned} H &:= \frac{1}{2}(\kappa_1 + \kappa_2) \\ K &:= \kappa_1 \kappa_2. \end{aligned}$$

Further, the Cayley–Hamilton theorem applied to S gives:

$$\text{III} - 2H\text{II} + K\text{I} = 0. \quad (1.1)$$

1.2. Lelievre’s Formula. Let us suppose that $K < 0$ and write $K = -1/\rho^2$. In this case, f admits asymptotic coordinates ξ, η , thus:

$$N_\xi \cdot f_\xi = 0 = N_\eta \cdot f_\eta.$$

It follows at once that there are functions a, b so that

$$a(N \times N_\xi) = f_\xi \quad b(N \times N_\eta) = f_\eta.$$

Now the symmetry of II : $N_\xi \cdot f_\eta = N_\eta \cdot f_\xi$, rapidly yields $a = -b$ while (1.1) evaluated on ∂_ξ gives $a^2 = \rho^2$ so that we have *Lelievre’s Formula*:

$$\begin{aligned} \rho(N \times N_\xi) &= f_\xi \\ \rho(N \times N_\eta) &= -f_\eta. \end{aligned} \quad (1.2)$$

Cross-differentiating (1.2) gives us two formulae for the tangential component of $f_{\xi\eta}$:

$$f_{\xi\eta}^T = \rho_\eta N \times N_\xi + \rho N \times N_{\xi\eta} = \rho_\xi N \times N_\eta - \rho N \times N_{\xi\eta}.$$

From this and the linear independence of $N \times N_\xi$ and $N \times N_\eta$, we easily see that $f_{\xi\eta}^T = 0$ if and only if ρ is constant if and only if $N \times N_{\xi\eta} = 0$, that is, $N : (\Sigma, \text{II}) \rightarrow S^2$ is a harmonic map. Finally, $f_{\xi\eta}^T = 0$ if and only if

$$(f_\xi \cdot f_\xi)_\eta = 0 = (f_\eta \cdot f_\eta)_\xi.$$

We conclude:

Theorem 1.1. *The following are equivalent:*

- K is constant.
- $N : (\Sigma, \text{II}) \rightarrow S^2$ is harmonic.
- Asymptotic coordinates can be chosen so that $\|f_\xi\| = 1 = \|f_\eta\|$. We say such coordinates are Tchebyshev for f .

1.3. Geometry of K -surfaces.

Definition. $f : \Sigma \rightarrow \mathbb{R}^3$ is a K -surface if it has constant, negative Gauss curvature.

From Theorem 1.1, we see that a K -surface admits Tchebyshev coordinates ξ, η with respect to which we have

$$\begin{aligned} \text{I} &= d\xi^2 + 2 \cos \omega d\xi d\eta + d\eta^2 \\ \text{II} &= \frac{2}{\rho} \sin \omega d\xi d\eta, \end{aligned}$$

where ω is the angle between the coordinate directions.

For such I, II , the Codazzi equations are vacuous while the Gauss equation reads

$$\omega_{\xi\eta} = \frac{1}{\rho^2} \sin \omega. \quad (1.3)$$

Thus any solution of (1.3) gives rise to a K -surface.

Let us now turn to the symmetries of the situation:

1.3.1. *Bäcklund transformations.* Let f be a K -surface and, following Bianchi (1879) and Bäcklund (1883), seek $\hat{f} : \Sigma \rightarrow \mathbb{R}^3$ such that:

- $\hat{f} - f$ is tangent to both f and \hat{f} .
- $\|\hat{f} - f\|$ is constant.
- $\hat{N} \cdot N$ is constant. (Bianchi considered the case $\hat{N} \cdot N = 0$.)

Then:

- (1) \hat{f} exists if and only if f is a K -surface (and then, by symmetry, \hat{f} is a K -surface too, in fact with the same value of K).
- (2) Given $a > 0$, $p_0 \in \Sigma$ and a ray $\ell_0 \subset T_{p_0}\Sigma$, one solves commuting ODE to get (locally) a unique \hat{f} with

$$\begin{aligned}\|\hat{f} - f\| &= \frac{2\rho}{a + a^{-1}} \\ \hat{N} \cdot N &= \frac{a^{-1} - a}{a^{-1} + a} \\ \hat{f}(p_0) &\in \ell_0.\end{aligned}$$

Thus, if $\hat{N} \cdot N = \cos \theta$, $\|\hat{f} - f\| = \rho \sin \theta$.

We write $\hat{f} = f_a$ and say that f_a is a *Bäcklund transform* of f ,

- (3) ξ, η are asymptotic, in fact Tchebyshev, for \hat{f} too. In classical terminology, f and \hat{f} are the focal surfaces of a W -congruence.
- (4) Permutability (Bianchi [5], 1892): given a K -surface f and two Bäcklund transforms f_a, f_b with $a \neq b$, one can choose initial conditions so that there is a fourth K -surface \hat{f} with

$$\hat{f} = (f_a)_b = (f_b)_a.$$

Exercise. \hat{f} is algebraic in f, f_a, f_b .

One can iterate the procedure and so build up a quad-graph of K -surfaces. At each point $p \in \Sigma$, the corresponding quad-graph of points in \mathbb{R}^3 is a discrete K -surface in the sense of Bobenko–Pinkall.

1.3.2. *Lie transform.* If ξ, η are Tchebyshev coordinates for a K -surface f , we have seen that the Gauss–Codazzi equations reduce to the sine-Gordon equation (1.3) for the angle ω between coordinate directions.

However, for $\mu \in \mathbb{R}^\times$, we observe that

$$\omega^\mu(\xi, \eta) := \omega(\mu^{-1}\xi, \mu\eta)$$

also solves (1.3) and so gives rise to a new K -surface f^μ with

$$I_{f^\mu} = d\xi^2 + 2\cos\omega^\mu d\xi d\eta + d\eta^2.$$

Such an f^μ is a *Lie transform* of f .

1.4. **Harmonic maps and flat connections.** We have seen that K -surfaces give rise to harmonic maps $(\Sigma, \mathbb{I}) \rightarrow S^2$. The converse is also true: let c be a conformal structure on Σ of signature $(1, 1)$, $*$ the Hodge-star of c and ξ, η null coordinates. Let $N : (\Sigma, c) \rightarrow S^2$ so that

$$*dN = N_\xi d\xi - N_\eta d\eta.$$

It is easy to see that N is harmonic if and only if

$$d(N \times *dN) = 0$$

in which case we can locally find $f : \Sigma \rightarrow \mathbb{R}^3$ such that $N \times *dN = df$, that is,

$$\begin{aligned}N \times N_\xi &= f_\xi \\ N \times N_\eta &= -f_\eta.\end{aligned}$$

It follows at once that, whenever f , equivalently N , immerses,

- (1) $N \perp f_\xi, f_\eta$ so that N is the Gauss map of f .
- (2) $N_\xi \cdot f_\xi = 0 = N_\eta \cdot f_\eta$ so that ξ, η are asymptotic for f whence $c = \langle \mathbb{I} \rangle$.

(3) $K = -1$ after recourse to (1.1).

1.4.1. *Flat connections.* The basic observation for all that follows is that harmonic maps give rise to a holomorphic family of flat connections on the trivial bundle $\underline{\mathbb{R}}^3 := \Sigma \times \mathbb{R}^3$. We rehearse this construction in such a way as to indicate how it generalises to any (pseudo)-Riemannian symmetric target.

So let $N : (\Sigma, c) \rightarrow S^2$ and $\rho^N : \Sigma \rightarrow \mathrm{O}(3)$ be the reflection across N^\perp . The orthogonal decomposition

$$\underline{\mathbb{R}}^3 = \langle N \rangle \oplus N^\perp$$

induces a decomposition of the flat connection d :

$$d = \mathcal{D} + \mathcal{N}$$

where N, ρ^N are \mathcal{D} -parallel and $\mathcal{N} \in \Omega^1(\mathfrak{so}(3))$ anti-commutes with ρ^N .

We shall several times have recourse to the identification $\mathbb{R}^3 \cong \mathfrak{so}(3)$ given by

$$v \mapsto (u \mapsto v \times u) \tag{1.4}$$

under which \mathcal{N} is identified with $N \times dN$.

The structure equations of the situation express the flatness of d and read:

$$\begin{aligned} R^{\mathcal{D}} + \tfrac{1}{2}[\mathcal{N} \wedge \mathcal{N}] &= 0 \\ d^{\mathcal{D}}\mathcal{N} &= 0, \end{aligned}$$

while N is harmonic if and only if $d * \mathcal{N} = 0$, or, equivalently, $d^{\mathcal{D}} * \mathcal{N} = 0$ since $[\mathcal{N} \wedge \mathcal{N}]$ is always zero.

Now write

$$\mathcal{N} = \mathcal{N}^+ + \mathcal{N}^-$$

where $*\mathcal{N}^\pm = \pm \mathcal{N}^\pm$. Then we have

$$\begin{aligned} d^{\mathcal{D}}\mathcal{N} &= d^{\mathcal{D}}\mathcal{N}^+ + d^{\mathcal{D}}\mathcal{N}^- = 0 \\ d^{\mathcal{D}} * \mathcal{N} &= d^{\mathcal{D}}\mathcal{N}^+ - d^{\mathcal{D}}\mathcal{N}^- \end{aligned}$$

so that N is harmonic if and only if $d^{\mathcal{D}}\mathcal{N}^\pm = 0$.

Let $\lambda \in \mathbb{C}^\times$ and define a connection d_λ on $\underline{\mathbb{C}}^3$ by

$$d_\lambda = \mathcal{D} + \lambda \mathcal{N}^+ + \lambda^{-1} \mathcal{N}^-.$$

Then, comparing coefficients of λ in R^{d_λ} , we have:

Proposition 1.2. *N is harmonic if and only if d_λ is flat for all $\lambda \in \mathbb{C}^\times$.*

We note that d_λ has the following four properties:

- (i) $\lambda \mapsto d_\lambda^+$ is holomorphic on \mathbb{C} with a simple pole at ∞ while $\lambda \mapsto d_\lambda^-$ is holomorphic on $\mathbb{C}^\times \cup \{\infty\}$ with a simple pole at 0.
- (ii) $\rho^N \cdot d_\lambda = d_{-\lambda}$. Here, and below, for connection D and gauge transformation $g : \Sigma \rightarrow \mathrm{O}(3, \mathbb{C})$, $g \cdot D = g \circ D \circ g^{-1}$, the usual action of gauge transformations on connections.
- (iii) $d_{\bar{\lambda}} = \overline{d_\lambda}$.
- (iv) $d_1 = d$.

Exercise. These properties uniquely determine d_λ .

Thus:

Proposition 1.3. *N is harmonic if and only if there is a family $\lambda \mapsto d_\lambda$ of flat connections with properties (i)–(iv).*

1.5. Spectral deformation. Let N be harmonic with flat connections d_λ . Since d_λ is flat, there is a locally a trivialising gauge $T_\lambda : \Sigma \rightarrow \text{SO}(3, \mathbb{C})$, that is,

$$T_\lambda \cdot d_\lambda = d.$$

Now fix $\mu \in \mathbb{R}^\times$ and set $d_\lambda^\mu := d_{\lambda\mu}$. We notice that $\lambda \mapsto d_\lambda^\mu$ has properties (i)–(iii) but $d_1^\mu = d_\mu$. It follows then that $\lambda \mapsto T_\mu \cdot d_\lambda^\mu$ has (i)–(iv) with respect to $N^\mu := T_\mu N : \Sigma \rightarrow S^2$. We therefore conclude

Theorem 1.4. $N^\mu : (\Sigma, c) \rightarrow S^2$ is harmonic and so gives rise to a K -surface f^μ .

Moreover,

$$dN^\mu = T_\mu(d_\mu N) = T_\mu(\mu d^+ N + \mu^{-1} d^- N).$$

If ξ, η are Tchebyshev for f , it follows from this that f^μ has first fundamental form

$$I_{f^\mu} = \mu^2 d\xi^2 + 2 \cos \omega d\xi d\eta + \mu^{-2} d\eta^2.$$

Thus $\hat{\xi} = \mu\xi$ and $\hat{\eta} = \mu^{-1}\eta$ are Tchebyshev for f^μ and the corresponding sine-Gordon solution is $\omega(\mu^{-1}\hat{\xi}, \mu\hat{\eta})$. Otherwise said, f^μ is a Lie transform of f .

1.6. Sym formula. Knowing the trivialising gauges T_λ , allows us to compute Lie transforms without integrations. Indeed, by definition,

$$T_\lambda \circ d_\lambda = d \circ T_\lambda$$

and differentiating this with respect to λ at μ yields:

$$\text{Ad}_{T_\mu}(\partial d_\lambda / \partial \lambda|_\mu) = d(\partial T_\lambda / \partial \lambda|_\mu T_\mu^{-1}).$$

The left side of this reads

$$\text{Ad}_{T_\mu}(\mathcal{N}^+ - \mathcal{N}^- / \mu^2) = \frac{1}{\mu} * \mathcal{N}^\mu,$$

or, using the identification (1.4),

$$N^\mu \times *dN^\mu = \mu d(\partial T_\lambda / \partial \lambda|_\mu T_\mu^{-1}).$$

Thus we have the *Sym formula* [27]:

$$f^\mu = \mu \partial T_\lambda / \partial \lambda|_\mu T_\mu^{-1}, \quad (1.5)$$

where we use (1.4) to view the right side as a map $\Sigma \rightarrow \mathbb{R}^3$. In particular, taking $\mu = 1$ and assuming, without loss of generality that $T_1 = 1$, we recover our original K -surface:

$$f = \partial T_\lambda / \partial \lambda|_{\lambda=1}. \quad (1.6)$$

1.7. Parallel sections and Bäcklund transformations. Again we start with a harmonic $N : (\Sigma, c) \rightarrow S^2$ and its family of flat connections d_λ . We seek to construct a holomorphic family of gauge transformations $r(\lambda) : \Sigma \rightarrow \text{SO}(3, \mathbb{C})$ so that the connections $r(\lambda) \cdot d_\lambda$ have properties (i)–(iv) with respect to a new map $\hat{N} : \Sigma \rightarrow S^2$. Since these gauged connections are flat, \hat{N} will be harmonic.

We will build our gauge transformations from d_λ -parallel subbundles of \mathbb{C}^3 using an avatar of a construction of Terng-Uhlenbeck [28]. First the algebra: for null line subbundles $L, L^* \leq \mathbb{C}^3$ with $L \cap L^* = \{0\}$, define

$$\Gamma_{L^*}^L(\lambda) = \begin{cases} \lambda & \text{on } L \\ 1 & \text{on } (L \oplus L^*)^\perp : \Sigma \rightarrow \text{SO}(3, \mathbb{C}). \\ \lambda^{-1} & \text{on } L^* \end{cases}$$

The decisive properties of $\Gamma_{L^*}^L$ is that it takes values in semisimple homomorphisms $\mathbb{C}^\times \rightarrow \text{SO}(3, \mathbb{C})$ and that $\text{Ad} \Gamma_{L^*}^L$ has only simple poles at 0 and ∞ .

Now fix $a > 0$ and choose L so that:

- (1) L is d_{ia} -parallel.
- (2) $\rho^N L = \bar{L}$. Denote this bundle by L^* .
- (3) $L \cap L^* = \{0\}$.

Remarks.

- (1) The last two conditions amount to demanding that $(L \oplus L^*)^\perp$ is a real line tangent to N or, equivalently, f . It is on this line that our Bäcklund transform will eventually lie.

- (2) The conditions are compatible: both $\rho^N L$ and \bar{L} are d_{-ia} -parallel and so coincide as soon as they do so at an initial point.
- (3) In fact, L is completely determined by the data of a unit tangent vector t at a single point $p_0 \in \Sigma$. We take for L_{p_0} and $L_{p_0}^*$ the $\pm i$ -eigenspaces of $v \mapsto t \times v$ and then define L and L^* by parallel transport. Of course, condition (3) may fail eventually.

Thanks to condition (2), we have

$$\begin{aligned} \rho^N \Gamma_{L^*}^L(\lambda) &= \Gamma_{L^*}^L(\lambda^{-1}) \\ \overline{\Gamma_{L^*}^L(\lambda)} &= \Gamma_{L^*}^L(1/\bar{\lambda}). \end{aligned} \tag{1.7}$$

After all this preparation, we finally set:

$$r(\lambda) = \Gamma_{L^*}^L \left(\left(\frac{1+ia}{1-ia} \right) \left(\frac{\lambda-ia}{\lambda+ia} \right) \right).$$

We have:

- $\lambda \mapsto r(\lambda)$ is holomorphic on $\mathbb{P}^1 \setminus \{\pm ia\}$.
- $\overline{r(\lambda)} = r(\bar{\lambda})$ so that, in particular, $r(\lambda)$ takes values in $\text{SO}(3)$ for $\lambda \in \mathbb{RP}^1$.
- $r(-\lambda) \circ \rho^N \circ r(\lambda)^{-1}$ is independent of λ and so coincides with $r(\infty) \rho^N r(\infty)^{-1} = \rho^{r(\infty)N}$. Thus

$$r(-\lambda) \circ \rho^N = \rho^{r(\infty)N} \circ r(\lambda). \tag{1.8}$$

- $r(1) = 1$.

We therefore set:

$$\begin{aligned} \hat{N} &= r(\infty)N \\ \hat{d}_\lambda &= r(\lambda) \cdot d_\lambda. \end{aligned}$$

Proposition 1.5. \hat{d}_λ has properties (i)–(iv) with respect to \hat{N} .

Proof. This is all a straightforward verification except for item (i). Since $r(\lambda)$ is holomorphic near zero and infinity, \hat{d}_λ has the same poles there as d_λ so the main issue is to see that $\lambda \mapsto \hat{d}_\lambda$ is holomorphic at $\pm ia$. This is where the fact that L and L^* are parallel comes in and is an immediate consequence of the following

Lemma 1.6. Let L, L^* be null line subbundles, $\lambda \mapsto d_\lambda$ any holomorphic family of connections and ψ_β^α any linear fractional transformation with a zero at α and a pole at β .

Then $\lambda \mapsto \Gamma_{L^*}^L(\psi_\beta^\alpha(\lambda)) \cdot d_\lambda$ is holomorphic at α if and only if L is d^α -parallel and holomorphic at β if and only if L^* is d_β -parallel.

□

We therefore conclude:

Theorem 1.7. $\hat{N} : (\Sigma, c) \rightarrow S^2$ is harmonic with associated flat connections $\hat{d}_\lambda = r(\lambda) \cdot d_\lambda$.

It is important that we have control on \hat{d}_λ since this allows us to iterate the construction as we shall see below.

Now let us turn to the geometry of the situation. N and \hat{N} are the Gauss maps of K -surfaces f and \hat{f} , both with $K = -1$. We compute \hat{f} via the Sym formula: if T_λ is a trivialising gauge for d_λ with $T_1 = 1$, then $T_\lambda r(\lambda)^{-1}$ is a trivialising gauge for \hat{d}_λ so that (1.6) yields

$$\hat{f} = f - \partial r / \partial \lambda|_{\lambda=1}.$$

The chain rule gives

$$\partial r / \partial \lambda|_{\lambda=1} = \frac{2ia}{1+a^2} (\Gamma_{L^*}^L)'(1)$$

which, under the identification (1.4), is

$$\frac{2a}{1+a^2} t$$

for t a real unit length section of $(L \oplus L^*)^\perp \leq df(T\Sigma)$.

We therefore conclude:

- $\hat{f} - f = -2t/(a + a^{-1})$ and so is tangent to f and of constant length.
- Since $r(\infty)$ is rotation about t through angle θ for

$$e^{i\theta} = \frac{1 + ia}{1 - ia},$$

$\hat{N} = r(\infty)N$ is orthogonal to t (so that $\hat{f} - f$ is tangent to \hat{f} as well) and

$$\hat{N} \cdot N = \operatorname{Re} \frac{1 + ia}{1 - ia} = \frac{a^{-1} - a}{a^{-1} + a}.$$

Thus $\hat{f} = f_a$, a Bäcklund transform of f .

- The conformal structures of Π and $\hat{\Pi}$ coincide (they are both c). Otherwise said, the asymptotic lines of f and \hat{f} coincide.

1.8. Permutability. Suppose that we have a K -surface and two Bäcklund transforms f_a and f_b produced from d_{ia} -parallel L_a and d_{ib} -parallel L_b , respectively. We now seek a fourth K -surface

$$f_{ab} = (f_a)_b = (f_b)_a.$$

For this we will need a d_{ib}^a -parallel \hat{L}_b and a d_{ia}^b -parallel \hat{L}_a . However, since $d_{ib}^a = r_a(ib) \cdot d_{ib}$ etc, we have natural candidates in

$$\begin{aligned} \hat{L}_b &:= r_a(ib)L_b \\ \hat{L}_a &:= r_b(ia)L_a. \end{aligned} \tag{1.9}$$

Exercise. Check that \hat{L}_a, \hat{L}_b so defined satisfy

$$\rho^a \hat{L}_b = \overline{\hat{L}_b} \quad \rho^b \hat{L}_a = \overline{\hat{L}_a}.$$

We therefore have K -surfaces $(f_a)_b$ and $(f_b)_a$ with associated connections $\hat{r}_b(\lambda) \cdot d_\lambda^a$ and $\hat{r}_a(\lambda) \cdot d_\lambda^b$. The key to showing these coincide is the following

Proposition 1.8. $\hat{r}_b r_a = \hat{r}_a r_b$.

For this we need a lemma which is a discrete version of Lemma 1.6:

Lemma 1.9. Let $\ell^\pm, \hat{\ell}^\pm$ be two pairs of distinct null lines, ψ_β^α a linear fraction transformation with a zero at α and a pole at β and $\lambda \mapsto E(\lambda)$ holomorphic near α and β . Then

$$\lambda \mapsto \Gamma_{\hat{\ell}^-}^{\hat{\ell}^+}(\psi_\beta^\alpha(\lambda))E(\lambda)(\Gamma_{\ell^-}^{\ell^+}(\psi_\beta^\alpha(\lambda)))^{-1}$$

is holomorphic at α if and only if $E(\alpha)\ell^+ = \hat{\ell}^+$ and holomorphic at β if and only if $E(\beta)\ell^- = \hat{\ell}^-$.

Proof of Proposition 1.8. We show that $R := \hat{r}_b r_a r_b^{-1} \hat{r}_a^{-1}$ is identically 1. Note R is holomorphic on $\mathbb{P}^1 \setminus \{\pm ia, \pm ib\}$ and $R(1) = 1$. Now Lemma 1.9 together with (1.9) shows that $\hat{r}_b r_a r_b^{-1}$ is holomorphic at $\pm ib$ and that $r_a r_b^{-1} \hat{r}_a^{-1}$ is holomorphic at $\pm ia$ so that R is holomorphic on \mathbb{P}^1 and so is constant. \square

In particular,

$$(f_a)_b = f - \partial(\hat{r}_b r_a)/\partial\lambda|_{\lambda=1} = f - \partial(\hat{r}_a r_b)/\partial\lambda|_{\lambda=1} = (f_b)_a$$

and we have established Bianchi permutability.

2. ISOTHERMIC SURFACES

2.1. Classical theory. First studied by Bour [8] in 1862, a surface $f : \Sigma \rightarrow \mathbb{R}^3$ is *isothermic* if it admits conformal curvature line coordinates x, y so that

$$\begin{aligned} I &:= e^{2u}(dx^2 + dy^2) \\ \Pi &= e^{2u}(\kappa_1 dx^2 + \kappa_2 dy^2). \end{aligned}$$

A more invariant formulation is that there should exist a non-zero holomorphic quadratic differential q on Σ such that

$$[q, \Pi] = 0,$$

or, more explicitly, $[S, Q] = 0$ where Q is the symmetric endomorphism with $q = I(Q, \cdot)$. The relationship between the two formulations is given by setting $z = x + iy$ and then $q = dz^2$.

Examples.

- cones, cylinders and surfaces of revolution are isothermic: for the last, parametrise the profile curve in the upper half plane by hyperbolic arc length to get conformal curvature line coordinates. In particular, we see that isothermic surfaces have no regularity.
- (Stereo-images of) surfaces of constant H in 3-dimensional space-forms. Here we take q to be the Hopf differential $\Pi^{2,0}$.
- quadrics. Sadly, I know no short or conceptual argument for this.

Isothermic surfaces have many symmetries:

- (1) Conformal invariance: if $\Phi : \mathbb{R}^3 \cup \{\infty\} \rightarrow \mathbb{R}^3 \cup \{\infty\}$ is a conformal diffeomorphism and f is isothermic, then $\Phi \circ f$ is isothermic too. This is because, while Π is certainly not conformally invariant, its trace-free part Π_0 is and $[q, \Pi] = [q, \Pi_0]$.
- (2) In 1867, Christoffel [17] showed that f is isothermic if and only if there is (locally) a *dual surface* $f^c : \Sigma \rightarrow \mathbb{R}^3$ such that
 - The metrics I and I^c are in the same conformal class.
 - f and f^c have parallel tangent planes: $df(T\Sigma) = df^c(T\Sigma)$.
 - $\det(df^{-1} \circ df^c) < 0$.

Of course, the symmetry of the conditions means that f^c is isothermic also and that $(f^c)^c = f$.

Examples.

- When f has constant mean curvature $H \neq 0$, $f^c = f + N/H$ which has the same constant mean curvature.
- When f is minimal, $f^c = N$, the Gauss map. Conversely, any conformal map $N : \Sigma \rightarrow S^2$ is isothermic with respect to any holomorphic quadratic differential q . Fixing such a q , we obtain a minimal surface $N^c : \Sigma \rightarrow \mathbb{R}^3$: this is the celebrated Weierstrass–Enneper formula!

- (3) After Darboux [21], we seek a surface $\hat{f} : \Sigma \rightarrow \mathbb{R}^3 \cup \{\infty\} = S^3$ such that

- f and \hat{f} induce the same conformal structure on Σ .
- f and \hat{f} have the same curvature lines.
- For each $p \in \Sigma$, there is a 2-sphere $S(p) \subset S^3$ to which both f and \hat{f} are tangent at p .

In classical terminology, f and \hat{f} are the enveloping surfaces of a *conformal Ribaucour sphere congruence*.

Here are the facts:

- (a) \hat{f} exists if and only if f is isothermic so that, by symmetry, \hat{f} is isothermic also.
- (b) For $a \in \mathbb{R}^\times$ and initial point $y_0 \in S^3 \setminus \{f(p_0)\}$, we can find a unique such \hat{f} with $\hat{f}(p_0) = y_0$ by solving a completely integrable 5×5 system of linear differential equations with a quadratic constraint (thus, to anticipate, finding a parallel section of a metric connection!).

We write $\hat{f} = f_a$ and call it a *Darboux transformation of f with parameter a* .

- (c) Permutability (Bianchi [3]): Given isothermic f and two Darboux transforms f_a and f_b with $a \neq b$, there is a fourth isothermic surface $f_{ab} = (f_a)_b = (f_b)_a$ which is algebraically determined by f, f_a, f_b . Indeed, Demoulin [22] shows that f, f_a, f_b, f_{ab} are pointwise concircular with constant cross-ratio $(f_b, f_a; f, f_{ab}) = a/b!$. We call a quadruple of surfaces related in this way a *Bianchi quadrilateral*.

One can iterate this procedure to construct a quad-graph of isothermic surfaces. At each point, the corresponding quad-graph of points in S^3 with concircular elementary quadrilaterals of prescribed cross-ratio gives a discrete isothermic surface in the sense of Bobenko–Pinkall [6] as we shall see in section 2.8.

Moreover, if we now add a third surface f_c , we can apply this result to obtain f_{ab}, f_{ac}, f_{bc} and then obtain an eighth surface f_{abc} such that

$$\begin{aligned} f_a, f_{ab}, f_{ac}, f_{abc} \\ f_b, f_{ab}, f_{bc}, f_{abc} \\ f_c, f_{ac}, f_{bc}, f_{abc} \end{aligned}$$

are *all* Bianchi quadrilaterals.

This “cube theorem”, also due to Bianchi (and also available for Bäcklund transformations of K -surfaces), can be viewed as the construction of a Darboux transform for discrete isothermic surfaces, see section 2.10.

- (4) Spectral deformation (Calapso 1903 [16], Bianchi 1905 [2]): Given f isothermic, there is a 1-parameter family f_t of isothermic surfaces with $f = f_0$ inducing the same conformal structure on Σ and having the same Π_0 . The f_t are called *T-transforms* of f .

Aside: We know that I and II determine a surface in \mathbb{R}^3 up to rigid motions. It is therefore natural to ask if the conformal invariants $\langle I \rangle$ and Π_0 determine a surface in S^3 up to conformal diffeomorphism. The T -transforms of an isothermic surface show that the answer is no but, according to Cartan¹, are the only such witnesses: if f is not isothermic it is determined up to conformal diffeomorphism by $\langle I \rangle$ and Π .

In this story, we recognise some familiar features: solutions in 1-parameter families and new solutions from commuting ODE. We will see how our gauge theoretic formalism applies in this situation.

2.2. Conformal geometry (rapid introduction). The conformal invariance of isothermic surfaces suggests that we should work on the conformal compactification $\mathbb{R}^3 \cup \{\infty\} = S^3$ of \mathbb{R}^3 . For this, Darboux [20] offers a convenient model which essentially linearises the situation.

Let $\mathbb{R}^{4,1}$ be a 5-dimensional Minkowski space with a metric (\cdot, \cdot) of signature $++++-$ and let $\mathcal{L} \subset \mathbb{R}^{4,1}$ be the light-cone:

$$\mathcal{L} = \{v \in \mathbb{R}^{4,1} : (v, v) = 0\}.$$

The collection of lines through zero in \mathcal{L} is the projective light-cone $\mathbb{P}(\mathcal{L})$ which is a smooth quadric in $\mathbb{P}(\mathbb{R}^{4,1})$ diffeomorphic to S^3 .

$\mathbb{P}(\mathcal{L})$ has a conformal structure: any section $\sigma : \mathbb{P}(\mathcal{L}) \rightarrow \mathcal{L}^\times$ of the projection $\mathcal{L}^\times \rightarrow \mathbb{P}(\mathcal{L})$ gives rise to a positive definite metric

$$g_\sigma(X, Y) := (d_X \sigma, d_Y \sigma)$$

and it is easy to see that $g_{e^u \sigma} = e^{2u} g_\sigma$. Conic sections give constant curvature metrics: more explicitly, let $t_0 \in \mathbb{R}^{4,1}$ have $(t_0, t_0) = -1$ and write $\mathbb{R}^{4,1} = \mathbb{R}^4 \oplus \langle t_0 \rangle$. Then the map $x \mapsto x + t_0 : S^3 \rightarrow \mathcal{L}$ from the unit sphere of \mathbb{R}^4 induces a conformal diffeomorphism onto $\mathbb{P}(\mathcal{L})$. Thus $\mathbb{P}(\mathcal{L}) \cong S^3$ as conformal manifolds.

k -spheres in S^3 are linear objects in this picture: they are the subsets $\mathbb{P}(W \cap \mathcal{L}) \subset \mathbb{P}(\mathcal{L})$ where $W \leq \mathbb{R}^{4,1}$ is a linear subspace of signature $(k+1, 1)$. For example, we obtain the circle through three distinct points in $\mathbb{P}(\mathcal{L})$ by taking W to be the $(2, 1)$ -plane they span.

Exercise. Let W be a $(3, 1)$ -plane. Show that reflection across W induces the inversion of $\mathbb{P}(\mathcal{L}) = S^3$ in the corresponding 2-sphere.

More generally, the subgroup $O_+(4, 1)$ of the orthogonal group that preserves the components of the light cone acts effectively by conformal diffeomorphisms on $\mathbb{P}(\mathcal{L})$ and, by the exercise, has all inversions in its image. It follows from a theorem of Liouville that $O_+(4, 1)$ is the conformal diffeomorphism group of S^3 .

In the light of all this, we henceforth treat maps $f : \Sigma \rightarrow \mathbb{P}(\mathcal{L})$ and identify such maps with null line subbundles $f \leq \mathbb{R}^{4,1}$ via $f(x) = f_x$.

¹See [11, 13] for modern treatments.

2.3. Isothermic surfaces reformulated. We give a third and final reformulation of the isothermic condition by exploiting the structure of S^3 as a homogeneous space for $G := O_+(4, 1)$.

Since G acts transitively, we have, for $x \in S^3$ an isomorphism

$$T_x S^3 \cong \mathfrak{g}/\mathfrak{p}_x,$$

where \mathfrak{p}_x is the infinitesimal stabiliser of x . The key algebraic ingredient in what follows is that \mathfrak{p}_x is a *parabolic* subalgebra with abelian nilradical: this means that the polar \mathfrak{p}_x^\perp with respect to the Killing form is an ad-nilpotent abelian subalgebra (in fact, it is the algebra of infinitesimal translations on the \mathbb{R}^3 obtained by stereoprojecting away from x).

Remark. We identify $\mathfrak{g} = \mathfrak{so}(4, 1)$ with $\wedge^2 \mathbb{R}^{4,1}$ via

$$(u \wedge v)w = (u, w)v - (v, w)u$$

and then $\mathfrak{p}_x^\perp = x \wedge x^\perp$.

Now $T_x^* S^3 \cong (\mathfrak{g}/\mathfrak{p}_x)^*$ which is isomorphic to $\mathfrak{p}_x^\perp \leq \mathfrak{g}$ via the Killing form. We have therefore identified $T^* S^3$ with a bundle of abelian subalgebras of \mathfrak{g} .

With this in hand, let q be a symmetric $(2, 0)$ -form on Σ and $f : \Sigma \rightarrow S^3$ an immersion. Then $q + \bar{q}$ is a section of $S^2 T^* \Sigma$ and so may be viewed as a 1-form with values in $T^* \Sigma$. Moreover, df and the conformal structure of S^3 allow us to view $T^* \Sigma$ as a subbundle of $f^{-1} T^* S^3$ and so as a subbundle of \mathfrak{g} . Chaining all this together, we see that $q + \bar{q}$ gives rise to a \mathfrak{g} -valued 1-form η taking values in the bundle of abelian subalgebras $\mathfrak{p}_f^\perp = f \wedge f^\perp$.

The crucial fact is now:

Proposition 2.1. *q is a holomorphic quadratic differential with $[q, \Pi_0] = 0$ if and only if $d\eta = 0$.*

The converse is also true and we arrive at our final formulation of the isothermic condition:

Theorem 2.2. *f is isothermic if and only if there is a non-zero $\eta \in \Omega^1(\mathfrak{g})$ with*

- (1) $d\eta = 0$.
- (2) η takes values in $f \wedge f^\perp$.

2.4. Flat connections. For f an isothermic surface with closed form η , we define, for each $t \in \mathbb{R}$, a metric connection $d_t = d + t\eta$ on $\underline{\mathbb{R}}^{4,1}$ and note that,

$$R^{d_t} = R^d + d\eta + \frac{1}{2}[\eta \wedge \eta] = 0$$

since each summand vanishes separately: $[\eta \wedge \eta] = 0$ since η takes values in the abelian subalgebra $f \wedge f^\perp$.

Thus we once again have a family of flat connections which we now exploit.

2.5. Spectral deformation. Since each d_t is flat, we may locally find a trivialising gauge $T_t : \Sigma \rightarrow SO(4, 1)$ with $T_t \cdot d_t = d$.

For $s \in \mathbb{R}^\times$, we have $d_{s+t} = d_s + t\eta$ so that

$$d_t^s := T_s \cdot d_{s+t} = d + \text{Ad}_{T_s} \eta$$

is flat for all t . Set $f^s = T_s f$ and $\eta_s := \text{Ad}_{T_s} \eta$ which takes values in the bundle of abelian subalgebras $f_s \wedge f_s^\perp$ so that $[\eta_s \wedge \eta_s] = 0$. The flatness of d_t^s now tells us that $d\eta_s = 0$ so that f_s is isothermic.

In fact, the f_s , $s \in \mathbb{R}^\times$, are the T -transforms of Bianchi and Calapso.

2.6. Parallel sections and Darboux transforms. Here the analysis is much easier than for K -surfaces: there we needed a slightly elaborate construction to arrive at a new surface from parallel line-bundles. For isothermic surfaces, the new surfaces *are* the parallel line bundles!

Indeed, for f isothermic with connections d_t and $a \in \mathbb{R}^\times$, choose a null line subbundle $\hat{f} \leq \underline{\mathbb{R}}^{4,1}$ such that

- (1) \hat{f} is d_a -parallel.

- (2) $f \cap \hat{f} = \{0\}$ (this condition may eventually fail far from the initial condition and this will introduce singularities into our transform).

Then:

- \hat{f} is isothermic.
- \hat{f} is a Darboux transform of f with parameter a .
- $\hat{d}_t = \Gamma_f^{\hat{f}}(1 - t/a) \cdot d_t$.

2.7. Permutability. Suppose now that we have isothermic f and two Darboux transforms f_a and f_b with connections d_t^a and d_t^b . We seek a fourth isothermic surface f_{ab} which is a simultaneous Darboux transform of f_a and f_b :

$$f_{ab} = (f_a)_b = (f_b)_a.$$

Thus we need $(f_a)_b$ to be d_b^a -parallel and $(f_b)_a$ to be d_a^b -parallel. The obvious candidates are:

$$\begin{aligned} (f_a)_b &= \Gamma_f^{f_a}(1 - b/a)f_b \\ (f_b)_a &= \Gamma_f^{f_b}(1 - a/b)f_a. \end{aligned}$$

We shall give two arguments that these coincide.

First note that, for $x, y, z \in \mathbb{P}(\mathcal{L})$,

$$t \mapsto \Gamma_y^x(t)z$$

is a rational parametrisation of the circle through x, y, z by $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ with

$$\begin{aligned} \infty &\mapsto x \\ 0 &\mapsto y \\ 1 &\mapsto z \end{aligned}$$

so that $\Gamma_y^x(t)z$ has cross-ratio t with x, y, z :

$$(x, y; z, \Gamma_y^x(t)z) = t.$$

We therefore conclude that

$$\begin{aligned} (f_a, f; f_b, (f_a)_b) &= 1 - b/a \\ (f_b, f, f_a, (f_b)_a) &= 1 - a/b \end{aligned} \tag{2.1}$$

whence a symmetry of the cross-ratio yields

$$(f_a, f; f_b, (f_a)_b) = (f_a, f; f_b, (f_b)_a)$$

so that $(f_a)_b = (f_b)_a$.

Our second argument extracts more. We prove

$$\Gamma_{f_a}^{(f_a)_b}(1 - t/b)\Gamma_f^{f_a}(1 - t/a) = \Gamma_{f_a}^{f_b}(\frac{1-t/b}{1-t/a}) = \Gamma_{f_b}^{(f_b)_a}(1 - t/a)\Gamma_f^{f_b}(1 - t/b). \tag{2.2}$$

Proof of (2.2). Let L and M denote the left and middle members of (2.2) respectively. It suffices to prove $L = M$ as the remaining equality follows by swapping the roles of a and b . For $L = M$, we note that L and M agree on f_a and f_a^\perp/f_a so, since both are orthogonal, it is enough to show that $Lf_b = f_b$ or, equivalently,

$$\Gamma_f^{f_a}(1 - t/a)f_b = \Gamma_{(f_a)_b}^{f_a}(1 - t/b)f_b.$$

However, these are rational parametrisations of the same circle that agree at $\infty, 0, b$ and so everywhere. \square

We now evaluate (2.2) on f to get

$$\Gamma_{f_a}^{(f_a)_b}(1 - t/b)f = \Gamma_{f_a}^{f_b}(\frac{1-t/b}{1-t/a})f = \Gamma_{f_b}^{(f_b)_a}(1 - t/a)f$$

and then take $t = \infty$ to conclude

$$(f_a)_b = \Gamma_{f_a}^{f_b}(a/b)f = (f_b)_a.$$

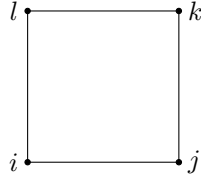
As a consequence $(f_b, f_a; f, f_{ab}) = a/b$ which is another version of (2.1).

The same argument proves the cube theorem: introduce a third Darboux transform f_c and so three surfaces f_{ab}, f_{bc}, f_{ac} . This gives rise to three new Bianchi quadrilaterals which we show share a common surface. To do this, evaluate (2.2) on f_c at $t = c$ to get

$$\Gamma_{f_a}^{f_{ab}}(1 - c/b)f_{ac} = \Gamma_{f_a}^{f_b}(\frac{1-c/b}{1-c/a})f_c = \Gamma_{f_b}^{f_{ab}}(1 - c/a)f_{bc}.$$

Here the left member is the simultaneous Darboux transform of f_{ab} and f_{ac} and the right is the simultaneous transform of f_{ab} and f_{bc} . The remaining equality follows by symmetry and the theorem is proved. As a bonus, we see that f_a, f_b, f_c, f_{abc} are also concircular with cross ratio $\frac{1-c/b}{1-c/a}$.

2.8. Discrete isothermic surfaces. View \mathbb{Z}^2 as the vertices of a combinatorial structure with edges between adjacent vertices and quadrilateral faces. We denote the directed edge from i to j by (j, i) and label the faces of an elementary quadrilateral as follows:



According to Bobenko–Pinkall [6], a discrete isothermic surface is a map $f : \mathbb{Z}^2 \rightarrow S^3 = \mathbb{P}(\mathcal{L})$ along with a *factorising function* a from undirected edges to \mathbb{R}^\times such that

- a is equal on opposite edges: $a(i, j) = a(l, k)$ and $a(i, l) = a(j, k)$.
- f has concircular values on each elementary quadrilateral with cross-ratio given by

$$(f(l), f(j); f(i), f(k)) = a(i, j)/a(i, l).$$

Thus the geometry is the pointwise geometry of Bianchi quadrilaterals of smooth isothermic surfaces.

2.9. Discrete gauge theory. The idea of discrete gauge theory is to replace connections by parallel transport and vanishing curvature by trivial holonomy. Here are the main ingredients:

- A *discrete vector bundle* V of rank n assigns a n -dimensional vector space V_i to each $i \in \mathbb{Z}^2$.
For example, the trivial bundle $\mathbb{R}^{4,1}$ has $\mathbb{R}_i^{4,1} = \mathbb{R}^{4,1}$ for each i .
- A *section* of V is a map $\sigma : \mathbb{Z}^2 \rightarrow \bigsqcup_i V_i$ such that $\sigma(i) \in V_i$, for all i .
- A *discrete connection* Γ on V , assigns to each directed edge (j, i) a linear isomorphism $\Gamma_{ji} : V_i \rightarrow V_j$ such that

$$\Gamma_{ij} = \Gamma_{ji}^{-1}.$$

Example: the trivial connection 1 on $\mathbb{R}^{4,1}$ has $1_{ji} = 1$, for all edges (j, i) .

- A section of V is *parallel* for Γ if $\sigma(j) = \Gamma_{ji}\sigma(i)$, for all edges (j, i) .
- A *discrete gauge transformation* assigns to each i a linear isomorphism $g(i) : V_i \rightarrow V_i$.
These act on connections by $(g \cdot \Gamma)_{ji} = g(j) \circ \Gamma_{ji} \circ g(i)^{-1}$.
- A connection Γ is *flat* if, on every elementary quadrilateral we have

$$\Gamma_{il}\Gamma_{lk}\Gamma_{kj}\Gamma_{ji} = 1$$

or, equivalently,

$$\Gamma_{kl}\Gamma_{li} = \Gamma_{kj}\Gamma_{ji}.$$

In this case, we can find a *trivialising gauge* $T : V \rightarrow \mathbb{R}^n$ such that

$$T \cdot \Gamma = 1,$$

that is, $T(i) : V_i \cong \mathbb{R}^n$ with

$$\Gamma_{ji} = T(j)^{-1}T(i).$$

We now have parallel sections through any point of V via $\sigma = T^{-1}x_0$ for constant $x_0 \in \mathbb{R}^n$.

2.10. Gauge theory of discrete isothermic surfaces. Given $f : \mathbb{Z}^2 \rightarrow S^3 = \mathbb{P}(\mathcal{L})$ and a factorising function a on edges, equal on opposite edges, we define a family of connections Γ^t on $\mathbb{R}^{4,1}$ by:

$$\Gamma_{ji}^t = \Gamma_{f(i)}^{f(j)}(1 - t/a(i, j)).$$

The arguments of section 2.7 and especially (2.2) essentially establish the following result:

Theorem 2.3. *f is discrete isothermic with factorising function a if and only if Γ^t is flat for all $t \in \mathbb{R}$.*

We may now apply all our previous gauge theoretic arguments in this new setting! We give just one example: a Darboux transform of a discrete isothermic surface should be given by a parallel null line subbundle. To verify this, fix $\hat{a} \in \mathbb{R}^\times$ and let $\hat{f} \leq \mathbb{R}^{4,1}$ be a null line subbundle such that

- (1) \hat{f} is $\Gamma^{\hat{a}}$ -parallel.
- (2) $f(i) \cap \hat{f} = \{0\}$, for all i .

Spelling out the parallel condition gives

$$\Gamma_{f(i)}^{f(j)}(1 - \hat{a}/a(i, j))\hat{f}(i) = \hat{f}(j)$$

so that $f(i), f(j), \hat{f}(i), \hat{f}(j)$ are concircular with fixed cross ratio $a(i, j)/\hat{a}$. Relabelling (2.2) to describe this quadrilateral gives

$$\Gamma_{f(j)}^{\hat{f}(j)}(1 - t/\hat{a})\Gamma_{f(i)}^{f(j)}(1 - t/a(i, j)) = \Gamma_{\hat{f}(i)}^{\hat{f}(j)}(1 - t/a(i, j))\Gamma_{f(i)}^{f(i)}(1 - t/\hat{a}).$$

Otherwise said:

$$\Gamma_{\hat{f}}^{\hat{f}}(1 - t/\hat{a}) \cdot \Gamma^t = \hat{\Gamma}^t.$$

Now $\hat{\Gamma}^t$ is flat for all t being a gauge of flat Γ^t so that \hat{f} is indeed isothermic with the same factorising function as f .

We remark that we can iterate this construction and so build up a map $F : \mathbb{Z}^3 \rightarrow S^3$ whose restrictions to level sets $\{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_3 = m\}$ are our iterated Darboux transforms. Moreover, the two other families of level sets obtained by holding n_1 or n_2 fixed also consist (as we have just seen) of concircular quadrilaterals with factorising cross-ratios and so are isothermic also. We therefore have a triple system of discrete isothermic surfaces!

Exercise. Find a spectral deformation for discrete isothermic surfaces.

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